(1)

Exercise 1.3.4

$\langle F, +, \cdot, ^{-1}, 0, 1 \rangle$

$P \subseteq F$

$x \leq y \Leftrightarrow x = y \vee y - x \in P$

Partial Ordered field axioms

PO-R $\quad s \leq s$

PO-A $\quad$ if $s \leq t \wedge t \leq s \Rightarrow s = t$

PO-T $\quad$ if $s \leq t \wedge t \leq u \Rightarrow s \leq u$

Total ordering

TO $\quad$ Either $s \leq t \vee s \geq t \vee$ both

Ordered field

(OF+) $\forall x \forall y \forall z [x \leq y \Rightarrow x + z \leq y + z]$

(OF·) $\forall x \forall y [x > 0 \wedge y > 0 \Rightarrow xy > 0]$

Prove F is an ordered field

1)

$$S \leq S$$

$$x = y \Rightarrow x \leq y \quad (\leq)$$

$$S = S \quad (\text{field axiom})$$

$$S + 0 = S$$

$\Rightarrow S = S \ (Id^{+})$ and also it's clear that $S = S$

2)

$$S \leq t \wedge S \geq t \Rightarrow S = t$$

both true

$$\boxed{S \leq t} \Rightarrow S = t \vee \boxed{S - t \in P}$$

$$\boxed{t \leq S} \Rightarrow S = t \vee \boxed{t - S \in P}$$

cannot both be true

$$S - t \in P \Rightarrow S - t \neq 0 \wedge -(S - t) \notin P$$

so $S - t \in P \Rightarrow t - S \notin P$

$\therefore$ $S = t \wedge t = S$ are the only sensible implications of $(S \leq t) \wedge (t \leq S)$

3)

$$s \leq t \wedge t \leq u \Rightarrow s \leq u$$

if both true, we get transitivity (OP-T) trivally

$$\boxed{x \leq y} \Leftrightarrow \boxed{x = y} \vee y - x \in P$$

both are true (assumptions)

$$\boxed{y \leq u} \Leftrightarrow \boxed{y = u} \vee u - y \in P$$

If $(x = y \wedge y = u)$

$\Rightarrow x = u$ (a number is equal to itself)

If $(x = y \wedge u - y \in P)$

$\Rightarrow u - x \in P$ (a number is eaual to itself)

If $(y - x \in P \wedge y = u)$

$\Rightarrow u - x \in P$ (a number is eaual to itself)

If $(y - x \in P \wedge u - y \in P)$

$\Rightarrow (x \leq y \wedge y \leq u)$ $(\leq)$

$\Rightarrow (y - x) \neq 0 \wedge (u - y) \neq 0$

$\Rightarrow -(y - x) \notin P \wedge -(u - y) \notin P$

* P is closed under addition and multiplication

$$(u + (-y)) + (y + (-x)) \in P$$

$$(u + (-y)) + ((-x) + y) \in P \quad (A+)$$

$$(u + (-x)) + ((-y) + y) \in P \quad (A+)$$

$$(u + (-x)) + 0 \in P \quad (\text{Inv}+)$$

$$u + (-x) \in P \quad (\text{Id}+)$$

$\Rightarrow$ If $(x \leq y \wedge y \leq u) \Rightarrow (x \leq u)$

(PO-T) holds

④

Show that P is precisely the set of positive elements

$x \in P$ if x satisfies only ONE of:

$x = 0$ $\quad x \in P$ $\quad (-x) \in P$

$\boxed{\begin{array}{c} x \text{ and } (-x) \text{ are opposite signs } \forall x \neq 0 \\ \text{assume } x > 0 \\ x + (-x) > (-x) \quad (OF+) \\ 0 > -x \quad (Inv +) \end{array}}$

$\downarrow$

$\therefore$ P is either the set of positives or the set of negatives

assume $x < 0 \wedge y < 0$ and P is the negatives

$(-x)(-y) > 0$ (proven above)

$-(x)(-y) > 0$ (proposition)

$-(-(x)(y)) > 0$ (proposition)

$\therefore x \cdot y > 0$, $\therefore x \cdot y \notin P$ if P was negative (contradiction)

$\therefore$ P is the positives by proof by contradiction

(5)

b) Let's prove that $1 > 0$

$1 \cdot 1 = 1$ (Id·)

$xy > 0 \ \forall x \forall y$ (proposition

$\therefore (1 \cdot 1) > 0$

$\therefore 1 > 0$

Let's prove that $x + z > x \ \forall (z > 0)$

Assume $z \geq 0$

$z + x \geq 0 + x$ (OF+)

$z + x \geq x$ (Id+)

if $z = 0$:

$0 + x \geq x$

$x \geq x \Rightarrow x = x$ (PO−R)

since the additive identity is uniqwe, $z > 0$ will never cause $x = x$

and thus $z + x > x \ \forall (z > 0)$

Then, $0 < 1 < 1 + 1 < (1 + 1) + 1 < ((1 + 1) + 1) + 1 \ldots$

And so, the ordering axioms imply an inifinite set?

6

$\mathbb{C}$

c.

$(z)(\bar{z}) = |z|^2$

PO-R	$s \leq s$
PO-A	$(s \leq t \wedge t \leq s) \Rightarrow s = t$
PO-T	$(s \leq t) \wedge (t \leq u) \Rightarrow s \leq u$
TO	$s \leq t \vee t \leq t \vee (s \leq t \wedge s \geq t)$
FO+	$x \leq y \Rightarrow x + z \leq y + z$
FO·	if $(x > 0 \wedge y > 0) \Rightarrow xy > 0$

→ what about $\mathbb{C}$ makes this impossible?

$\underbrace{\sqrt{-1} \cdot \sqrt{-1} = -1}$ (an axiom of $\mathbb{C}$?)

FO· is violated

prove $-1 < 0$:

prove $1 > 0$:

$1 \cdot 1 = 1$ (Id·)

$x \cdot x > 0 \quad \forall (x \neq 0)$ (proposition)

$\therefore 1 > 0$

prove x and $-x$ are opposite signs:

assume $x > 0$

then $x > x + (-x)$ (OF+)

$x + (-x) > x + (-x) + (-x)$

$0 > -x$ (Inv+)